Multilayer network valuation under bail-in

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Abstract

We propose a model of simultaneous valuation of different classes of financial instruments in a financial system. We build on the literature on financial contagion using models of cross-holdings of equity participations and debt in different seniority classes. We combine these with recently proposed methods of network valuation under stochastic external assets, resulting in a multilayer network valuation model, where each layer has its own valuation function, reflecting the payoff structure of the financial instruments in that layer. We extend this model to include bail-ins and contingent convertible debt instruments, two recently introduced mechanisms for recapitalizing banks at the brink of failure without the use of public bail-outs. We provide a Matlab implementation of this extension.

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Contents

Introduction

The 2007-2009 global financial crisis exemplified how interconnectedness of financial institutions, henceforth banks for simplicity, combined with a low loss-absorption capacity of banks can propagate distress in the financial system (Laeven and Valencia 2013). This low loss-absorption capacity led to capital shortfalls at many banks when the crisis hit, forcing policy makers to step in with public bail-outs in order to prevent the default of major banks. The unintended consequence of these emergency measures was an increase in sovereign credit risk, creating a vicious circle between the creditworthiness of banks and governments (Fratzscher and Rieth 2015). Bail-outs further create a moral hazard problem by rewarding less prudent banks with an implicit insurance (Dam and Koetter 2012). Those insights have motivated the adoption of alternative approaches for dealing with capital shortfalls at banks, aimed at breaking the sovereign-bank nexus and addressing moral hazard.

In 2014, the Single Resolution Mechanism was created (BRRD 2014, SRMR 2014), intended to act as one of the three pillars of a banking union to be established, alongside the Single Supervisory Mechanism (SSM 2013; EBA 2013) and the European Deposit Insurance Scheme (not yet implemented). Resolution aims to provide an alternative to insolvency for the orderly exit of a bank from the market, while ensuring the provision of critical functions. It would start only after recovery measures have failed and the bank is considered to be failing or likely to fail (BRRD 2014). One tool available to resolution authorities is bail-in, which involves the full or partial write-down or conversion into equity of entire seniority classes of liabilities. Similarly, in the US, the resolution mechanism for systemically important institutions (SIFI's) is specified in the Orderly Liquidation Authority (OLA) of Title II of the Dodd-Frank Wall Street Reform and Consumer Protection Act of 2010 (Reform and Act 2010), including the possibility of bail-in. It differs from the European Single Resolution Mechanism in that the resolution (and bail-in) is designed to be applied after the failure of the institution (McAndrews et al. 2014[1]). We will present a framework modelled after the EU regulatory landscape in this paper; however, adapting it to the US Dodd-Frank act regulation, or any other regulation for resolution that includes the concept of bail-in, should be straightforward, mainly changing the parameters for resolution and recapitalization levels.

From a policy maker's perspective, the bail-in mechanism has the advantage of offering a recapitalization option for failing banks that does not rely on recourse to public funds. However, it implies that the losses that are otherwise shifted to the taxpayer will be borne within the system where the instruments subject to a bail-in are being held, hence creating a potential channel for contagion. The phenomenon of financial contagion has been studied before the crisis (Eisenberg and Noe 2001), but has received far more attention from both the academic as well as the regulatory community since (Upper 2011). Various approaches have been proposed to study contagion due to bilateral loans (Elsinger, Lehar, and Summer 2006), equity participations (Elsinger 2009) and some types of derivatives (Fischer 2014). Bail-in, being a relatively new mechanism, has received less attention so far. Most notably, Hüser et al. 2017 study the potential scope of direct losses from a data set on the European banking system.

Another measure for dealing with capital shortfalls at banks and preventing bail-out is the recognition of contingent convertible debt instruments (CoCo's), thereby essentially creating this new asset class (Flannery 2016). These instruments are similar to convertible bonds, with the main difference being that the conversion to equity (or write-down) is triggered by a pre-defined set of conditions rather than discretionary. CoCo's have been succesfully adopted in the past by banks such as Lloyd's bank, Credit Suisse and Rabobank (De Spiegeleer and Schoutens 2011).

The contribution of the present work is to introduce bail-in mechanisms and CoCo's into formal models of financial contagion. As noted by Fischer 2014, such a model

[1]It is not clear what actions would be adopted according to US regulation if a SIFI were considered to be close to failing. McAndrews et al. 2014, p.5, express their opinion that the regulatory entities would only act upon failure.

constitutes an extension of the classical framework of Merton 1974 to account for cross-holdings of equities and liabilities. Our model extends the model of Fischer 2014 and the related model of Elsinger 2009 by including the possibility of a bail-in mechanism and CoCo's, and ex-ante valuation as introduced by Barucca et al. 2016. It extends the model of Hüser et al. 2017 by including higher-order contagion effects from cascades and the aforementioned extensions, and generalizes it to allow for different allocations of equity between bailed-in creditors and old equity owners. The potential scope of applications of the model includes central banks, where it can serve as an add-on to stress test exercises, as well as regulators who have to decide on capital buffers for systemically important institutions and for systemic risk (CRD-IV 2013). For this purpose we provide Matlab code freely available under the MIT licence.

Literature Review

In order to quantify contagion in financial networks due to interbank liabilities, it is necessary to develop a model that captures the nonlinear behaviour of the propagation of risk in a stressed scenario. Within this stream of research, Eisenberg and Noe 2001 describe a financial network mathematically and provide an algorithm to compute a clearing payment vector in such a model. The clearing payment vector consists of the payments made by each bank to other banks in the network due to claims, taking into account the possibility of default of a bank. Elsinger 2009 extend Eisenberg and Noe 2001's model to include the possibility of cross-ownership of equity in the network, and to allow for multiple seniority classes of liabilities. This extension is necessary to incorporate the bail-in regulation. Hence, the model used in this project is largely based on Elsinger 2009's model, applied to the simulation of bail-in regulation. Rogers and Veraart 2013 introduce bankruptcy costs in the Eisenberg and Noe 2001-model, and argue that with bankruptcy costs, solvent banks may have incentives to rescue failing banks in the network. Besides the Eisenberg and Noe 2001 model and its extensions, other simpler models have been proposed to quantify contagion effects. Furfine 2003 consider a model in which the recovery rate after the default of a bank is predetermined and equal for all banks, instead of being determined endogenously as in Eisenberg and Noe 2001's model.

The aforementioned network models are a tool that can be applied to real world data to assess the systemic risk in a particular network of banks. Elsinger, Lehar, and Summer 2006 model a network of Austrian banks using Eisenberg and Noe 2001' framework, with data from the Austrian Central Bank. They conclude that in this particular network, the correlation of banks' assets contributes more to systemic risk than the contagion due to interbank liabilities, which has rare effects. Upper and Worms 2004 do a similar empirical study for German banks, concluding that the contagion effects of the default of a bank in the network can have considerable effects. This discrepancy between empirical studies suggests that contagion effects are highly dependent on the network topology.

When applying network models to quantify systemic risk, it is common to follow a stress test approach in which initially an external shock affects the network, and then the contagion propagates endogenously. Two main approaches can be applied for the initial shock: assuming that it affects only one bank, such as in traditional stress testing, or assuming that the external shock (such as a macroeconomic stress scenario) affects the whole network at once. Of particular relevance is the case of an external shock to certain assets which are held by multiple banks in the network (overlapping portfolios). Beltran, Cordell, and Thomas 2017 observe that during the 2007-2009 financial crisis, the liquidity and price of Collateral Debt Obligations of Asset Backed Securities (ABS CDO's) dropped substantially, having caused \$218 billions in losses to global banks, insurance companies and asset managers until January 2019. Moreover, Krishnamurthy 2010 argue that the liquidity crisis affected many other assets unrelated to the "toxic" ABS CDO's. Caccioli et al. 2015 develop a stress test model in which overlapping portfolios of banks are subject to the depreciation of one "toxic" asset, and followed by an endogenous contagion as described in Furfine 2003's model. They conclude that the combination of overlapping portfolios and contagion due to interbank liabilities causes far more systemic risk than any of these sources of risk considered individually.

Another topic of interest when modelling systemic risk in a financial network are fire sales. In a stressed scenario, some banks which are at or close to default may engage in fire sales of their assets, to meet leverage targets, to raise liquidity or as part of their resolution.Cifuentes, Ferrucci, and Shin 2005 incorporate fire sales into the Eisenberg and

Noe 2001-model, by assuming that banks sell part of there assets as soon as they breach a capital adequacy constraint given by the ratio of equity to total assets. The price of the assets is then endogenously generated as a function of supply (fire sales) and demand. Siebenbrunner, Sigmund, and Kerbl 2017 propose a similar fire sales model, however assuming that banks sell their assets at the event of default.

In the context of stress testing, the Eisenberg and Noe 2001-model and its extensions are used for macroprudential stress tests. These are a class of stress tests which consider the system-wide effects of a stressed macroeconomic scenario, including endogenous interactions caused by interbank liabilities, overlapping portfolios, and fire sales. The Bank of England has developed the RAMSI stress test (Burrows et al. 2012, Aikman et al. 2009), which includes the Eisenberg and Noe 2001-model to quantify contagion after the macroeconomic shock, as well as fire sales and funding costs. Aymanns et al. 2017 discuss the developments in computational models of financial stability and their applications to macroprudential stress tests.

The Elsinger 2009-model and its extensions can be used in the context of ex-ante valuation as an extension of Merton 1974's structural approach for pricing debt. Suzuki 2002 and Fischer 2014 develop such a network valuation model with multiple seniorities, and calculate the price of debt in this framework. Barucca et al. 2016 add to this model the concept of a "local ex-ante valuation", in which banks only have information about their own counterparties. We add to this literature the ex-ante network valuation of debt with bail-in and the ex-ante network valuation of CoCo's. In particular, the network valuation of CoCo's extends the previous models of pricing CoCo's within Merton's structural approach, to account for the network structure (see Pennacchi 2010).

Chapter 1

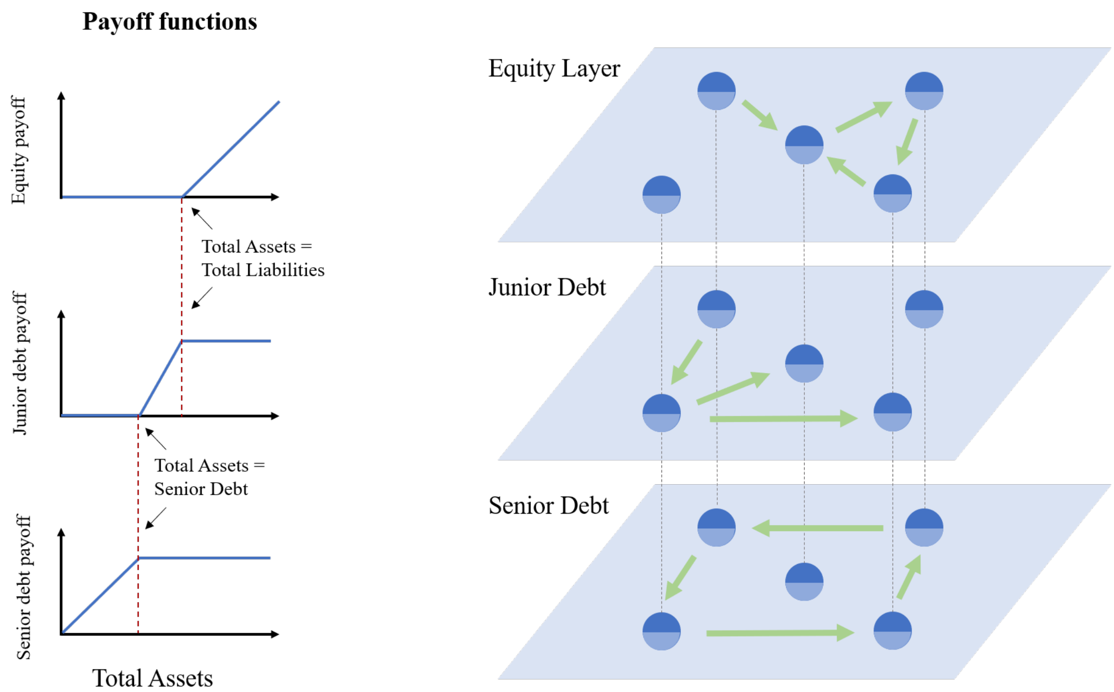
The Interbank Network Model

In this section we present the formal framework that will be used for modelling bail-in in section 2. The framework builds on Elsinger 2009, which is an extension of the frameworks of Eisenberg and Noe 2001 and Elsinger, Lehar, and Summer 2006, and we will largely follow their notation here.

We consider a system of $n - 1$ financial entities, and a "sink node" with label n corresponding to external creditors towards which the financial entities have liabilities. In the context of financial systems, the financial entities may be thought of as banks or other financial institutions, and the external creditors could be other banks not included in the system of consideration, corporations or individual depositors. It is worth pointing out, however, that the formal framework per se is not limited to this interpretation. Extensions to include corporations or even households are a matter of data availability and the aim of the analysis, as the model framework is agnostic to these differences. For simplicity, and keeping the aim of this study in mind, we will continue to refer to the entities in the system as banks, however. They are represented via a stylized balance sheet, which on the asset side consists of external assets as well as assets that represent either claims on or equity participations in other entities in the system. On the liability side, we distinguish several seniority classes of liabilities, going to entities within the system and outside, and equity, which is the residual quantity. The set of n financial entities and the different types of connections between them constitute a multilayer network. This corresponds to the model of Elsinger 2009, which we extend in the subsequent sections.

Before we begin the formal introduction of the model, we will lay its main logic qualitatively. Figure 1.1 shows the different layers of the network model, in this case a simple example with five banks and two seniority layers of liabilities. As can be seen, the same set of five banks have different links in each of the layers. Each of these links is valued using a different payoff function, depending on the layer. The payoff for the equity layer looks like the payoff for a long call option with strike price equal to the total liabilities. The payoff for the most junior layer can be described as long call option with strike equal to the value of senior liabilities plus a short call option with strike equal to total liabilities. Payoffs for more senior equity classes can be described analogously, with strikes equal to the combined value of the more senior liability classes (or 0 for the most senior class) in the long position and equal to this value plus the value of the current class for the short position. The reminiscence of the model of Merton 1974 here is not misleading, and will be discussed in greater detail in section 4.

Figure 1.1: Payoff functions for multiple layers of cross-holdings



1.1 Definition of a financial system

Definition 1.1.1 (External assets)**.** *Let $e_i \geq 0$ be the price of all external assets of bank i. The vector $e \in \mathbb{R}^n$ is the vector of external assets.*

Definition 1.1.2 (Liabilities)**.** *Let $L_{i,j,s}$ denote the liabilities from bank i towards bank j within seniority class $s \in \{1, \ldots, S\}$, S being the most junior seniority class. We assume that the "sink node" has no liabilities, i.e. $L_{n,j,s} = 0$ for all j, s. The corresponding tensor of liabilities is $L \in \mathbb{R}^{n \times n \times S}$.*

Definition 1.1.3 (Ownership)**.** *It is possible that bank i owns a share $\Theta_{j,i}$ of bank j. The ownership matrix Θ is required to be a holding matrix, as defined in Elsinger 2009, i.e it must satisfy that $\sum_i \Theta_{j,i} \leq 1$ for all j, and, for any $S \subset \{1, \ldots, n\}$, $\sum_{i \in S} \Theta_{j,i} < 1$.*

We say that a bank is in default when its liabilities are greater than its assets. A bank is bailed-in when it is considered to be failing-or-likely-to fail (FLTF) by the regulators. We discuss the modelling of the FLTF decision and bail-in in section 2.

Hence, the financial system is fully described by the vector of assets $e \in \mathbb{R}^n$, the liabilities tensor $L \in \mathbb{R}^{n \times n \times S}$, and the ownership matrix $\Theta \in [0, 1]^{n \times n}$. It is not trivial to determine the payments between banks if some bank is in default. The clearing payment matrix is defined as follows:

Definition 1.1.4 (Clearing payment matrix)**.** *A clearing payment matrix is a matrix $P \in \mathbb{R}^{n \times S}$ of total payments made by each bank, so that the payments respect the following criteria:*

- *limited liability: the total payments of each bank must not exceed the total assets of the bank.*
- *priority of debt claims: the bank's stockholders receive no value unless all liabilities are repaid fully.*
- *seniority hierarchy of debt claims: lower seniority classes receive no payoff unless all liabilities of higher seniority are repaid fully.*
- *proportionality: in case of default, all creditors of the same seniority class are paid proportionally to the liabilities against them.*

1.2 Calculation of the clearing payment vector

Consider first the case with no seniority structure ($S = 1$). Let $\bar{p} \in \mathbb{R}^n$ be the vector of total liabilities of each bank, $\bar{p}_i = \sum_j L_{i,j}$. Define the matrix $\Pi \in \mathbb{R}^{n \times n}$ of relative liabilities by

$$\Pi_{i,j} = \begin{cases} \frac{L_{i,j}}{\bar{p}_i}, & \text{if } \bar{p}_i > 0 \\ 0, & \text{if } \bar{p}_i = 0 \end{cases}$$

In order to determine the clearing payment vector, it is first necessary to define the concept of the equity of a bank. The equity of a bank consists of its total amount of assets (external assets plus assets from interbank lending and holdings) minus its total liabilities, $\bar{p}_i$. In the case that the holdings matrix $\Theta = 0$, the equity vector is simply expressed as:

$$V^*(p) = (Assets - Liabilities)^+ = (e + \Pi' p - \bar{p})^+$$

This definition corresponds to the value of equity for the owner (equity payoff), which is non-negative. We will also consider the related definition of accounting value of equity (which can be negative), which we will also refer to as equity:

$$Equity = Assets - Liabilities = e + \Pi' p - \bar{p}$$

If the holdings matrix Θ is not null, the equity is defined as follows.

Definition 1.2.1 (Equity)**.** *Given a financial system with no seniority structure, and a payment vector $p \in \mathbb{R}^n$, the vector $V^*(p) \in \mathbb{R}^n$ is an equity vector if and only if it is a fixed point of the following map:*

$$\psi(V) = (e + \Pi' p - \bar{p} + \Theta' V)^+$$

So that,

$$V^*(p) = (e + \Pi' p - \bar{p} + \Theta' V^*(p))^+ \tag{1.1}$$

Elsinger 2009 (lemma 5, Appendix) show that the map has a unique fix point $V^*(p)$ for any $p \in \mathbb{R}^n$, provided that Θ is a holding matrix.

To be consistent with the clearing criteria, a payment vector must satisfy:

$$p_i = \begin{cases} 0, & \text{if } (e + \Pi' p + \Theta' V^*(p))_i \leq 0 \\ (e + \Pi' p + \Theta' V^*(p))_i, & \text{if } 0 \leq (e + \Pi' p + \Theta' V^*(p))_i \leq \bar{p}_i \\ \bar{p}_i, & \text{if } \bar{p}_i \leq (e + \Pi' p + \Theta' V^*(p))_i \end{cases}$$

Elsinger 2009 prove the existence of a clearing payment vector under this framework and discuss conditions for its uniqueness. Of more interest from a practical perspective it to calculate the (largest) clearing vector. First, it is necessary to define a new variable $W^*(p)$ given a payment vector $p \in [0, \bar{p}]$, such that $W^*(p)$ is a fixed point of the map $\psi^W(W) = e + \Pi' p - \bar{p} + \Theta'(W \vee 0)$.

Lemma 1.2.1. *[Elsinger 2009] The sequence W^k defined by $W^0(p) = e + \Pi' p - \bar{p}$ and $W^{k+1}(p) = e + \Pi' p - \bar{p} + \Theta' \Lambda^k W^{k+1}$, where $\Lambda^k = diag(W^k > \boldsymbol{0})$ converges to the largest fixed point of ψ^W.*

Theorem 1.2.1. *[Elsinger 2009] If Θ is a holding matrix, the sequence $p^{i+1} = (W^*(p^i) + \bar{p})^+ \wedge \bar{p}$, with $p^0 = \bar{p}$, converges to the largest clearing payment vector.*

1.3 Seniority structure

The previous results show how to calculate a clearing payment vector for a financial system in the particular case that there is no seniority structure. Consider now the case of a financial system with seniority structure ($S > 1$). In this case, a payment matrix P is determined by its value in each seniority class: $P \in \mathbb{R}^{n \times S}$, and the total liabilities are $\bar{P} \in \mathbb{R}^{n \times S}$ where $\bar{P}_{i,s} = \sum_j L_{i,j,s}$. Similarly, the tensor of relative liabilities is defined for each seniority class:

$$\Pi_{i,j,s} = \begin{cases} \frac{L_{i,j,s}}{\bar{P}_{i,s}}, & \text{if } \bar{P}_{i,s} > 0 \\ 0, & \text{if } \bar{P}_{i,s} = 0 \end{cases}$$

Definition 1.3.1 (Equity Vector)**.** *The equity vector under a given payment matrix is defined by the equation:*

$$V^*(P) = \Big(e + \sum_{s=1}^{S} \Pi'_{\cdot,\cdot,s}\, P_{\cdot s} - \sum_{s=1}^{S} \bar{P}_{\cdot s} + \Theta' V^*(P)\Big)^+ \tag{1.2}$$

It is a fixed point of the map

$$\psi^S(V) = \Big(e + \sum_{s=1}^{S} \Pi'_{\cdot,\cdot,s}\, P_{\cdot,s} - \sum_{s=1}^{S} \bar{P}_{\cdot,s} + \Theta' V\Big)^+ \tag{1.3}$$

A clearing payment matrix must respect the seniority structure. Elsinger 2009 provide a characterization for a clearing payment matrix in this case:

Definition 1.3.2. *[Clearing payment matrix] A matrix P^* satisfying $P^*_{i,j} \in [0, \bar{P}_{i,j}]$, $\forall i, j$ is a clearing payment matrix if and only if it is a fixed point of the map*

$$\Phi^S(P)_{i,T} = \begin{cases} \bar{P}_{i,T}, & \text{if } e_i + \mathcal{A}_{i,T} \geq \bar{P}_{i,T} \\ (e_i + \mathcal{A}_{i,T})^+, & \text{otherwise} \end{cases} \tag{1.4}$$

Where $\mathcal{A}_{\cdot,T} = \sum_{s=1}^{S} \Pi'_{\cdot,\cdot,s} P_{\cdot,s} - \sum_{s=1}^{T-1} \bar{P}_{\cdot,s} + \Theta' V^*(P)$.
Hence, $\forall T \in \{1, \ldots, S\}$,

$$P^*_{\cdot,T} = \Big(e + \sum_{s=1}^{S} \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} - \sum_{s=1}^{T-1} \bar{P}_{\cdot,s} + \Theta' V^*(P^*)\Big)^+ \wedge\ \bar{P}_{\cdot,T} \tag{1.5}$$

Intuitively, a bank will only pay liabilities for a certain seniority class T if all seniority classes $s < T$ have been fully paid.

Elsinger 2009 define a sequential clearing procedure for calculating the clearing payment matrix, starting at the most junior seniority class assuming that all senior seniority classes have been fully paid, and proceeding iteratively to more senior classes if the bank is insolvent. Formally, following the notation in Elsinger 2009, define $H = (H_1, \ldots, H_n)$ to be the vector of seniority classes in which the current iteration is, with $H^0 = (S, \ldots, S)$. Define

$$e_i^H = e_i + \sum_{j=1}^{n} \sum_{s=1}^{H_j - 1} \Pi_{j,i,s} \bar{P}_{j,s} - \sum_{s=1}^{H_i - 1} \bar{P}_{i,s} \tag{1.6}$$

Which corresponds to the assets of bank i in the current iteration H. Similarly, define

$$\Pi^H_{i,j} = \Pi_{i,j,H(i)}, \quad (p_H)_i = P_{i,H(i)} \tag{1.7}$$

Start the iteration with $H^0 = (S, \ldots, S)$, and let $p^*_{H_0}$ be the clearing vector for the financial system with no seniority structure $(e^{H^0}, \Pi^{H^0}, \bar{p}_{H^0}, \Theta)$. If the equity of all banks in this financial system is non-negative, or the only banks with negative equity have reached the most senior seniority structure in the iteration, the procedure is completed. Otherwise, let $\Lambda = diag(e^{H^0} + \Pi^{H^0\prime} p^*_{H_0} - \bar{p}_{H^0} + \Theta' V^*(p^*_{H_0}) < 0)$, and set $H^1 = H^0 - 1'\Lambda$. Iterating this procedure leads to a clearing matrix in finite steps (Elsinger 2009).

Chapter 2

Modelling Bail-in

To model bail-in, we assume that the K most junior seniority classes are bail-in-able. If a certain amount $\text{Bail-In}_{i,j,s}$ of the liabilities $L_{i,j,s}$ are to be converted to equity, bank j will gain a share $C_{i,j,s}$ of bank i. This share can be dependent on the bail-in amount $\text{Bail-In}_{i,j,s}$ and the equity values of the banks, $\text{Equity}_j^n = \left(e + (\Theta^n)' V^*(P^*) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} - \sum_s \bar{P}^n_{\cdot,s}\right)_j$, so we model the conversion share as a function:

$$C : \mathbb{R}^{n \times n \times S} \times \mathbb{R}^n \to \mathbb{R}^{n \times n \times S} \tag{2.1}$$

$$(\text{Bail-In}, \text{Equity}) \mapsto C\,(\text{Bail-In}, \text{Equity}) \tag{2.2}$$

Clearly, $C_{i,j,s} = 0$ must hold for $s \leq S - K$. We also impose that $C(0, \text{Equity}) = 0$ for all values of Equity. For convenience of notation, we will usually refer to $C\,(\text{Bail-In}, \text{Equity})$ as simply the conversion matrix C. In order for the updated holdings matrix to still be consistent with the definition of a holdings matrix (definition 1.1.3), we have to make one additional assumption on the conversion matrix:

Assumption 2.0.1 (Conversion matrix)**.** *The shares gained in each bank sum to less than one:*

$$\sum_s \sum_j C_{i,j,s} < 1 \;\; \forall i \tag{2.3}$$

Note that while the bail-in is overall neutral for the investors of the bailed in bank, it could imply mutually offsetting gains and losses for either the old equity owners or the bailed in creditors, depending on the conversion factors C. A too low conversion factor (or a complete write-down of the liabilities, corresponding to a conversion factor of 0) would mean a transfer of wealth from the bailed in creditors to the old equity owners of the bank, and vice versa for too high conversion factors.

In practice, conversion factors would have to be determined from applicable regulations or covenants. In the absence of such data, we present here what we consider a canonical choice for the conversion factor, the *fair share*, which ensures that the bail-in is wealth-neutral for all affected investors:

Definition 2.0.1 (Fair conversion matrix)**.** *A conversion matrix is said to be fair if the share gained by the bailed in creditors is equal to the share of the bailed in liabilities in the new equity, consisting of the bailed in liabilities plus the old equity, in the case the latter is positive. Let* $\textit{Bail-In}^n_{j,s} = \bar{P}^n_{j,s} - \bar{P}^{n+1}_{j,s}$ *and* $\textit{Bail-In}^n_j = \sum_{s=1}^{S} \bar{P}^n_{j,s} - \bar{P}^{n+1}_{j,s}$ *denote the amount of liabilities currently bailed in at bank* j *over one seniority class and all seniority classes, respectively, and let* $\textit{Equity}^n_j = \left(e + (\Theta^n)' V^*(P^*) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} - \sum_s \bar{P}^n_{\cdot,s}\right)_j$ *denote the current equity of bank* j *(the reasons for the choice of notation will become clear in section 2.2). Then the fair conversion factor for creditor* i *in seniority class* s *is given by:*

$$C^{fair}_{j,i,s} = \begin{cases} y_{j,i,s} & \text{if } \textit{Equity}^n_j \leq 0 \\ \frac{\Pi_{j,i,s} \textit{Bail-In}^n_{j,s}}{\left(\textit{Equity}^n_j + \textit{Bail-In}^n_j\right)} & \text{otherwise} \end{cases} \tag{2.4}$$

where $y_{j,i,s}$, $\sum_{i,s} y_{j,i,s} \leq 1$, *is a share that remains to be determined otherwise, in practice likely by computing a ratio similar to the above using paid-in capital instead of current book value equity. While it would seem consistent with the idea of the fair share that the bailed in creditors would gain full control of the bailed in bank, we have to assume that the* $y_{j,i,s}$ *are such that* C^{fair} *is still consistent with assumption 2.0.1 (i.e.* $\sum_{i,s} y_{j,i,s} < 1$*). We thus propose the fair choice of* y *as:*

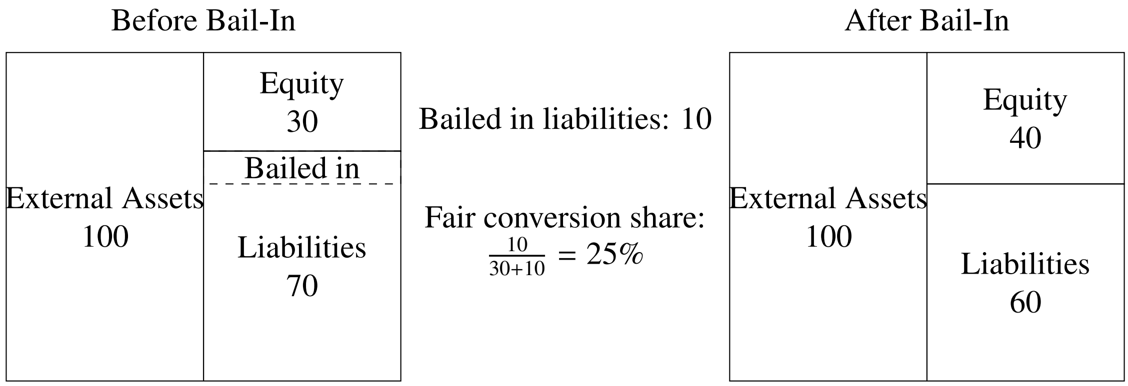
$$y^{fair}_{j,i,s} = \begin{cases} 0 & \text{if } \textit{Bail-In}^n_{j,s} = 0 \\ \gamma \frac{\Pi_{j,i,s} \textit{Bail-In}^n_{j,s}}{\textit{Bail-In}^n_j} & \text{otherwise} \end{cases} \tag{2.5}$$

where $\gamma \in [0,1)$ *ensures* $\sum_{i,s} y_{j,i,s} < 1$. *In order to be able to ensure that the non-positive wealth impact of the fair conversion matrix also holds for the old equity owners, we further define* $C^{fair,\, y=1}$, *which explicitly violates assumption 2.0.1 (i.e.* $\sum_{i,s} y_{j,i,s} = 1$*), and will be used later:*

$$C^{fair,\, y=1}_{j,i,s} = \begin{cases} 0 & \text{if } Bail\text{-}In^n_{j,s} = 0 \\ \frac{\Pi_{j,i,s} Bail\text{-}In^n_{j,s}}{\left(\max\left(0, Equity^n_j\right) + Bail\text{-}In^n_j\right)} & \text{otherwise} \end{cases} \tag{2.6}$$

We will see later (section 2.2) that if $C^{\text{fair}}_{j,i,s} = C^{\text{fair},\, y=1}_{j,i,s}$, the bail-in is neutral for all affected creditors, and deal with the general case there as well. For now we present a simple example to demonstrate the mechanics of a bail-in. Consider a bank with only external assets of 100 monetary units, only one creditor in one seniority class of liabilities totalling 70 monetary units, of which 10 are bailed in (figure 2.1). The fair share in this example is given by 25%, leaving the old equity owners with an unchanged equity stake of $(1-25\%)*40 = 30$ and the bailed in creditor with a $25\%*40 = 10$ equity stake, equal to the amount of bailed in liabilities. A higher conversion share, e.g. 50% would imply equal equity stakes of 20 for both investors, thus transferring 10 monetary units from the old shareholder to the creditor. A lower share, e.g. 0%, corresponding to a complete write-down for the creditor, would imply a transfer of 10 units from the creditor to the old shareholder. Note that the structure of the balance sheet, as shown in the right part of figure 2.1, would look the same under all conversion shares, as these only affect the investors.

Figure 2.1: Fair conversion share example



We now briefly describe the relevant bail-in regulation, which will help us model the bail-in threshold and relate the abstract notations in the previous section to the bail-in mechanism.

2.1 Legal framework

The EU legal framework for bail-in is established in the SRM regulation [1] (see article 27), and discussed in Hüser et al. 2017. In this framework, the SRM has the capability of writing down or converting into equity liabilities from a bank that is failing-or-likely-to fail (FLTF). The regulation follows the approach that all liabilities are bail-inable unless they are excluded in the regulation. Secured liabilities, deposits, and short term interbank liabilities are not bail-in-able. For an exhaustive list of all liabilities that are not bail-in-able, refer to the SRM regulation, article 27 (3).

2.1.1 Conditions for resolution and recapitalization

A bank must be considered to be failing-or-likely-to fail (FLTF) for a resolution. EU authorities have not specified a quantitative threshold for this. A conservative estimation is 7% of CET1 capital to risk-weighted assets (Hüser et al. 2017 infer this estimation from EU legislation).

The recapitalization level reached after the resolution is also not quantified by EU authorities, however it is stated in the European Banking Authority (EBA) Regulatory Technical Standards (RTS) that after the resolution, a bank should have a capitalization which is similar to the average in a defined per group (Hüser et al. 2017). Following this guideline, Hüser et al. 2017 determine a level of recapitalization of CET1 10.5% for the so-called Significant Institutions under direct supervision of the Single Supervisory Mechanism (SSM).

Basel III capital regulations know two types of basic capital regulations: leverage-based and risk-weighted. Roughly speaking, both define that capital, under some definition, needs to be a minimum share of assets, taken either at book values or as a risk

[1] https://eur-lex.europa.eu/legal-content/EN/TXT/PDF/?uri=CELEX:32014R0806from=EN

exposure amount, respectively. In order to account for risk-weights, we define a bank-specific weighting function $W_i\left(e_i, \left(\sum_s \Pi'_{\cdot,\cdot,s} P_{\cdot,s}\right)_i, (\Theta' V^*(P))_i\right)$ that returns either the simple sum of its inputs in the case of a leverage ratio or the risk exposure amount in the case of a risk-weighted ratio. The capitalization of a bank is then given by:

$$\frac{W_i\left(e_i, \left(\sum_s \Pi'_{\cdot,\cdot,s} P_{\cdot,s}\right)_i, (\Theta' V^*(P))_i\right) - \bar{P}_i}{W_i\left(e_i, \left(\sum_s \Pi'_{\cdot,\cdot,s} P_{\cdot,s}\right)_i, (\Theta' V^*(P))_i\right)} \tag{2.7}$$

For simplicity we will assume a leverage-based threshold in the remainder of the paper. The extension to a risk-weighted threshold, or to a set of multiple thresholds including both risk-weighted and leverage-based definitions, is straightforward.

2.1.2 Structure of a bank's balance sheet

The bail-in must respect the seniority structure of the bank, considering only the most junior liabilities. For any practical implementation it is thus important to map the the different classes of liabilities to the seniority structure of the model. While our framework is not restricted to banks, as discussed in section 1, we provide such a mapping of seniority classes for bank liabilities here, keeping in mind the intended scope of applications of the model. Following Hüser et al. 2017, we specify the following types of assets and seniority classes of capital and liabilities.

Capital and Liabilities

- *CET1*: Common Equity Tier 1 Capital. Consists of the bank's core capital, mainly common shares and retained earnings (Committee et al. 2010).
- *T2*: Tier 2 Capital. Largely consisting of capital reserves, e.g. from revaluations. We will map CET1+T2 to the equity class in the formal model.

Liabilities, ordered by seniority (junior to senior):

- *AT1*: Additional Tier 1 Capital. Consists mainly of preferred stock and contingent convertibles debt instruments (CoCo's). While we would classify preferred stock as equity, AT1 can be dominated by CoCo's, and we will focus on the latter here and classify AT1 as the most junior liability class. In fact, the hybrid nature of

CoCo's warrants special treatment in the context of the network model, which we will discuss in section 3.

- *Subordinated unsecured debt issued*: Junior class of unsecured wholesale funding.
- *Senior unsecured debt issued*: Senior class of unsecured wholesale funding.
- *Deposits*: Deposits by bank customers. Not bail-inable.
- *Secured debt issued*: Collateralized whosesale funding. These creditors have claims to particular assets on the balance sheet, hence their claims on the assets have the highest priority. Not bail-inable.

Assets

- *Interbank Debt Holdings*: Interbank lendings.
- *Interbank Equity Holdings*: Participations in other banks in the network.
- *External Assets*: All other assets.

Table 2.2 provides a mapping of the legal framework to the formal model presented in section 1:

2.1.3 Description of the bail-in procedure

The following algorithm presents the main procedure used in the analysis:

- *Step 1*: Loss on assets, given by the possible bail-in or default of other banks in the network, and the corresponding loss of interbank debt holdings and interbank equity holdings.
- *Step 2*: Bail-in. If the capital of a bank is lower than a required bank-specific fraction $\lambda_i^B \geq 0$ of total assets, the liabilities are bailed-in by order of seniority. The amount of liabilities that are bailed in is given by the necessary amount of equity to meet the bank-specific recapitalization level $\lambda_i^R \geq \lambda_i^B$. Note that this

Figure 2.2: Balance sheet of a bank, mapped to the network model.

Interbank Loans $\sum_s \Pi'_{\cdot,\cdot,s} P_{\cdot,s}$	Secured Debt Issued $\bar{P}_{\cdot,1}$	Not bail-inable
Interbank Equity Holdings $\Theta' V^*(P)$	Deposits $\bar{P}_{\cdot,2}$	
Other Assets e	Senior Unsecured Debt $\bar{P}_{\cdot,3}$	
	Subordinated Unsecured Debt $\bar{P}_{\cdot,4}$	
	AT1 $\bar{P}_{\cdot,5}$	CoCo's (section 3)
	Capital (CET1 + T2) $V^*(P)$	

is a generalization of the standard models presented in section 1, which can be recovered by setting $\lambda_i^R = \lambda_i^B = 0$ for all i. If not all liabilities within a seniority level must be converted, a partial bail-in will be effectuated.

2.2 Calculation of the clearing payment matrix

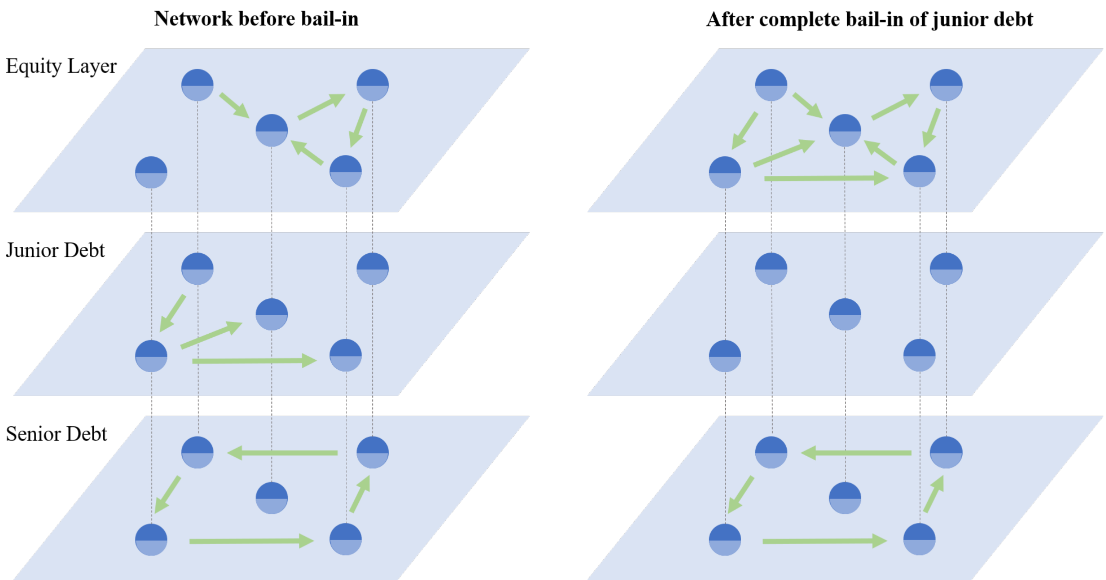
The clearing payment matrix in this framework is defined as follows:

Definition 2.2.1 (Clearing payment matrix with bail-in)**.** *A matrix $P^* \in \mathbb{R}^{n \times S}$ is a clearing payment matrix with bail-in if and only if it is a fixed point of the map*

$$\Phi^B(P)_{i,T} = \begin{cases} \bar{P}^B_{i,T}, & \text{if } e_i + \mathcal{A}^B_{i,T} \geq \bar{P}^B_{i,T} \\ (e_i + \mathcal{A}^B_{i,T})^+, & \text{otherwise} \end{cases} \tag{2.8}$$

where

Figure 2.3: Network reconfiguration effect of a bail-in



$$\mathcal{A}^B_{\cdot,T} = \sum_{s=1}^{S} \Pi'_{\cdot,\cdot,s} P_{\cdot,s} - \sum_{s=1}^{T-1} \bar{P}^B_{\cdot,\cdot,s} + (\Theta^B)' V^*(P, \bar{P}^B, \Theta^B) \tag{2.9}$$

Note that a bail-in implies a reconfiguration of both the liability as well as the ownership network. Figure 2.3 shows a graphical representation of the network reconfiguration effects implied by such a bail-in in the simple case of only two seniority classes of debt and a full bail-in (with positive conversion) junior debt of all banks who have liabilities in this seniority class: as can be seen, all links are deleted from the fully bailed in layer and added to the equity holdings layer. We refer to the total liabilities matrix resulting from this operation as $\bar{P}^B$, and to the adjusted holdings matrix as Θ^B. Note that the payoff functions, as illustrated in figure 1.1, do not change under this operation – the bailed in liabilities will be subject to the equity payoff function instead of the payoff function of their original seniority class.

$\bar{P}^B$ and Θ^B are calculated together with the clearing payment matrix with bail-in using the following procedure:

1. (*Initialization*) Start the iteration with $\bar{P}^1 = \bar{P}$, $\Theta^1 = \Theta$.

2. (*Begin Do-While*) Compute P^* according to the Elsinger algorithm using $\bar{P}^n$ and Θ^n.

3. Check which banks have breached the bail-in threshold by computing the corresponding capital ratio,

$$\Lambda^n = diag\left(\frac{e + (\Theta^n)'V^*(P^*) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} - \sum_s \bar{P}^n_{\cdot,s}}{e + (\Theta^n)'V^*(P^*) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s}} < \lambda_B\right) \tag{2.10}$$

4. Compute the amount of required bail-in under P^*, $\bar{P}^n$ and Θ^n for each bank. The bail-in amount is capped at the amount of bail-in-able liabilities, and we further do not allow negative bail-ins (lowering of the capital ratio for banks that are above the threshold). We thus obtain the vector Bail-In^n of bail-in amounts at each bank:

$$\text{Bail-In}^n = \Lambda^n \min\left(\sum_{s=S-K+1}^{S} \bar{P}^n_{\cdot,s}, \left(\sum_s \bar{P}^n_{\cdot,s} - \left(1-\lambda^R\right)\left(e + (\Theta^n)'V^*(P^*) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s}\right)\right)^+\right) \tag{2.11}$$

5. Update the liabilities and holdings matrices:

$$\bar{P}^{n+1}_{\cdot,s} = \begin{cases} \bar{P}^n_{\cdot,s} & \text{if Bail-In}^n \leq \sum_{k>s} \bar{P}^n_{\cdot,k} \\ \bar{P}^n_{\cdot,s} - \left(\text{Bail-In}^n - \sum_{k>s} \bar{P}^n_{\cdot,k}\right) & \text{if } \sum_{k>s} \bar{P}^n_{\cdot,k} \leq \text{Bail-In}^n < \sum_{k\geq s} \bar{P}^n_{\cdot,k} \\ 0 & \text{otherwise} \end{cases} \tag{2.12}$$

The updated holding matrix follows from $\bar{P}^{n+1}$. Define Bail-In$^n_{i,j,s}$ to be the liabilities of bank i bailed in in seniority class s, $f^n_{i,j,s}$ to be the share bank j gains of bank i due to the bail-in of the liabilities $L_{i,j,s}$, and $\bar{f}^n$ to be the vector of total shares gained over each bank during the bail-in in iteration n,

$$\text{Bail-In}^n_{i,j,s} = \Pi_{i,j,s}\left(\bar{P}^n_{i,s} - \bar{P}^{n+1}_{i,s}\right) \tag{2.13}$$

$$f^n_{i,j,s} = C_{i,j,s}\left(\text{Bail-In}^n, \text{Equity}^n\right) \tag{2.14}$$

$$\bar{f}^n_i = \sum_{j,s} f^n_{i,j,s} \tag{2.15}$$

As alluded to in definition 2.0.1, even when a fair conversion matrix is used, the conversion may lead to a gain for the old equity owners if the equity was negative prior to the bail-in and positive afterwards. In order to make the impact of the bail-in non-positive for all investors we define a penalty $f_i^p = \sum_{j,s}\left(C_{i,j,s}^{\text{fair}, y=1} - C_{i,j,s}\right)$, by which the equity shares of the old owners will be decreased. The updated conversion matrix is then given by:

$$\Theta_{j,i}^{n+1} = \left(1 - \bar{f}_j^n - \mathbb{I}_{\bar{f}_j^n > 0} f_j^p\right)\Theta_{j,i}^n + \sum_s f_{j,i,s}^n, \quad \forall i, j \tag{2.16}$$

where $\mathbb{I}$ represents an indicator function. Note that the penalty is optional and f_j^p may be set to zero if a non-positive bail-in (see definition 2.2.2 below) is not required.

6. (*Termination condition*) If $\bar{P}^{n+1} = \bar{P}^n$ and $\Theta^{n+1} = \Theta^n$, or $\bar{P}_{\cdot,s}^{n+1} = 0 \;\forall\; s > S - K$ terminate and set $\bar{P}^B = \bar{P}^{n+1}, \Theta^B = \Theta^{n+1}$, otherwise continue at step 2.

We now establish a useful characterization of bail-ins:

Definition 2.2.2 (Neutral and non-positive bail-ins)**.** *We say that a bail-in is neutral if it does not change the payoff value of all investments (debt plus equity) for all investors:*

$$\forall\, i, j :\; \Theta_{j,i}^{n+1} \max\left(0, \mathit{Equity}_j^n + \mathit{Bail\text{-}In}_j^n\right) - \Theta_{j,i}^n \max\left(0, \mathit{Equity}_j^n\right) = \sum_s \Pi_{j,i,s} \mathit{Bail\text{-}In}_{j,s}^n \tag{2.17}$$

Note that this relates to both bailed-in creditors as well as diluted equity investors. Non-positive bail-ins are defined analogously:

$$\forall\, i, j :\; \Theta_{j,i}^{n+1} \max\left(0, \mathit{Equity}_j^n + \mathit{Bail\text{-}In}_j^n\right) - \Theta_{j,i}^n \max\left(0, \mathit{Equity}_j^n\right) \le \sum_s \Pi_{j,i,s} \mathit{Bail\text{-}In}_{j,s}^n \tag{2.18}$$

Lemma 2.2.1.

(i) When $\mathit{Equity}_j^n + \mathit{Bail\text{-}In}_j^n \le 0$, any bail-in is non-positive.

(ii) When $Equity^n_j \leq 0, Equity^n_j + Bail\text{-}In^n_j \geq 0$ *and* $f^p_i = \sum_{j,s}\left(C^{fair,\, y=1}_{i,j,s} - C_{i,j,s}\right), \forall i$*, a fair conversion matrix implies a non-positive bail-in.*

(iii) When $Equity^n_j \geq 0$ *and* $f^p_i = \sum_{j,s}\left(C^{fair,\, y=1}_{i,j,s} - C_{i,j,s}\right), \forall i$ *, a fair conversion matrix implies a neutral bail-in.*

Proof. See Appendix A. □

Lemma 2.2.2.

If the bail-in is neutral and $Equity^n + Bail\text{-}In^n \geq 0$*, then*

$$V^*(\bar{P}^{n+1}, \Theta^{n+1}) = Equity^n + Bail\text{-}In^n \tag{2.19}$$

Proof. Consider the equity map $\psi^S(V, P) = \left(e + \sum_{s=1}^S \Pi'_{\cdot,\cdot,s} P_{\cdot,s} - \sum_{s=1}^S \bar{P}_{\cdot,s} + \Theta' V\right)^+$ defined in section 1.3. Assume the clearing payment matrix under $\bar{P}^{n+1}$ and Θ^{n+1} is $\bar{P}^{n+1}$, so that all banks are solvent. If, under this assumption, $\text{Equity}^{n+1} \geq 0$, then the assumption is validated, due to the definition of a clearing payment matrix 1.3.2. Applying definition 2.2.2, we obtain for all j,

$$\psi^S(\text{Equity}^n + \text{Bail-In}^n, \bar{P}^{n+1})_j = \tag{2.20}$$

$$= \left(e_j + \sum_s \sum_i \Pi_{i,j,s} \bar{P}^{n+1}_{i,s} - \sum_s \bar{P}^{n+1}_{j,s} + \sum_i \Theta^{n+1}_{i,j}(\text{Equity}^n_i + Bail\text{-}In^n_i)\right)^+ \tag{2.21}$$

$$= \left(e_j + \sum_s \sum_i \Pi_{i,j,s} \bar{P}^{n+1}_{i,s} - \sum_s \bar{P}^{n+1}_{j,s} + \sum_i \left(\Theta^n_{i,j} V^*(\bar{P}^n, \Theta^n)_i + \sum_s \Pi_{i,j,s} Bail\text{-}In^n_{i,s}\right)\right)^+ \tag{2.22}$$

$$= \left(e_j + \sum_s \sum_i \Pi_{i,j,s}(\bar{P}^{n+1}_{i,s} + Bail\text{-}In^n_{i,s}) - \sum_s \bar{P}^n_{j,s} + Bail\text{-}In^n_j + \sum_i \Theta^n_{i,j} V^*(\bar{P}^n, \Theta^n)_i\right)^+ \tag{2.23}$$

$$= \text{Equity}^n_j + Bail\text{-}In^n_j \geq 0 \tag{2.24}$$

This proves that $\text{Equity}^n + \text{Bail-In}^n$ is a fixed point of $\psi^S(V)$. Elsinger 2009 prove that such a solution is unique (see Lemma 5). Hence, $V^*(\bar{P}^{n+1}, \Theta^{n+1}) = \text{Equity}^{n+1} = \text{Equity}^n + \text{Bail-In}^n \geq 0$, and the clearing payment matrix is $\bar{P}^{n+1}$.

□

Theorem 2.2.1.

(i) The sequence $\bar{P}^n$ converges to a limit $\bar{P}^B$ for any conversion matrix.

(ii) If the conversion is neutral, under $\bar{P}^B$ every bailed-in bank either reaches its recapitalization ratio or fully bails in all bail-in-able liability classes.

Proof.

(i) Note that by definition, $\bar{P}^{n+1} \leq \bar{P}^n$ and further note that $\bar{P}^n \geq \mathbf{0}$ for all n, where the inequalities are understood to be component-wise. Hence there exists a monotone limit $\lim_{n\to\infty} \bar{P}^n = \bar{P}^B$, in the norm $\|A\|_\infty = \sup_{i,j} |A_{i,j}|$.

(ii) Recall that the bail-in amount Bail-In^n is set as

$$\text{Bail-In}^n = \min\left(\sum_{s=S-K+1}^{S} \bar{P}^n_{\cdot,s}, \left(\sum_s \bar{P}^n_{\cdot,s} - (1-\lambda^R)\Big(e + (\Theta^n)' V^*(\bar{P}^n, \Theta^n) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} \Big) \right)^+ \right) \tag{2.25}$$

If $\text{Bail-In}^n_j = \left(\sum_{s=S-K+1}^{S} \bar{P}^n_{\cdot,s}\right)_j$, the bank fully bails in all bail-in-able liability classes. If $\text{Bail-In}^n_j = 0$, there is no bail-in. Otherwise,

$$\text{Bail-In}^n_j = \left(\sum_s \bar{P}^n_{\cdot,s} - (1-\lambda^R)\Big(e + (\Theta^n)' V^*(\bar{P}^n, \Theta^n) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} \Big) \right)_j \tag{2.26}$$

By Lemma 2.2.2,

$$\begin{aligned} V^*(\bar{P}^{n+1}, \Theta^{n+1})_j &= \text{Equity}^n_j + \text{Bail-In}^n_j && (2.27) \\ &= \lambda^R_j \left(e + (\Theta^n)' V^*(\bar{P}^n, \Theta^n) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} \right)_j \geq 0 && (2.28) \\ & && (2.29) \end{aligned}$$

The new capital ratio is:

$$\lambda_j = \frac{V^*(\bar{P}^{n+1}, \Theta^{n+1})_j}{V^*(\bar{P}^{n+1}, \Theta^{n+1})_j + \sum_s \bar{P}^{n+1}_{j,s}} \tag{2.30}$$

$$= \frac{\lambda_j^R \left(e + (\Theta^n)' V^*(\bar{P}^n, \Theta^n) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s}\right)_j}{\text{Equity}_j^n + \text{Bail-In}_j^n + \sum_s \bar{P}^n_{j,s} - \text{Bail-In}_j^n} \tag{2.31}$$

$$= \lambda_j^R \tag{2.32}$$

Hence, the recapitalization ratio has been reached. In fact, this argument shows that the iteration ends after one step, after which every bailed-in bank either reaches its recapitalization ratio or fully bails in all bail-in-able liability classes, and every non bailed-in bank maintains its original capital structure, having a capital ratio higher than the bail-in ratio.

□

The matrix Θ^B is defined from $\bar{P}^B$ analogously as Θ^{n+1} is defined from $\bar{P}^{n+1}$.

Note that the above results show that there exist ways to minimize losses due to bail-ins, and make them even perfectly neutral under some circumstances. Alternative specifications, including a complete or partial write-down of loans as assumed in the model of Hüser et al. 2017, would be feasible, however they would imply a wealth transfer from bailed in creditors to old owners of the bailed in bank. This seems critical when seen in the light of the '*no-creditor-worse-off*'-principle (NCWO), foreseen both in EU and US regulations (BRRD 2014, Reform and Act 2010), which states that no creditor should be worse off under a bail-in than they would be under insolvency. Since the Elsinger algorithm without bail-in can be seen as an analogue to insolvency, it seems that a fair conversion would be a necessary requirement in order to satisfy NCWO, a question we will leave to legal scholars.

Chapter 3

CoCo's

Contingent convertibles (CoCo's) are another source of contagion in the interbank network. They have a similar role as bail-in, however their trigger is generally not based on a regulatory decision, but specified in the contract. In this section we describe how to include CoCo's in the framework described in the previous section 2. Many details of the modelling of CoCo's will be omitted, due to the similarity with the case of bail-in. Instead, we will focus on the particularities of CoCo's. The main differences between CoCo's and bail-in within the context of our model is that the former may have a separate conversion trigger from the latter, and that there is no recapitalization target. Note that bail-in and CoCo's are two separate, but mutually compatible additions to the standard model framework presented in section 1, and may also be added to it in isolation.

As mentioned previously, the main particularity of CoCo's is their conversion trigger, which is defined in the contract. The following are the main types of triggers (De Spiegeleer and Schoutens 2011):

- **Market Trigger**: The trigger may be based on the market price of shares or CDS spreads of the issuing institution. The advantage of this type of trigger is that it is forward looking. However, there could be price manipulation and speculation once the share (or CDS) price gets close to the trigger (De Spiegeleer and Schoutens 2011).
- **Accounting Trigger**: This type of trigger is based on accounting ratios such as the Tier 1 common capital ratio as defined in Basel III. The disadvantage of this

trigger is that it is based on historical values and updated periodically. In fact, many of the financial institutions that were rescued during the financial crisis had capital ratios above the minimum requirement (Kuritzkes and Scott 2009).

- **Regulatory Trigger**: Corresponds to the decision of a relevant regulation authority, such as in the case of bail-in.

In the framework described in sections 1 and 2, only the accounting trigger can be modelled. However, this is a fairly general case, as the main examples of adoption of CoCo's are based on accounting triggers (De Spiegeleer and Schoutens 2011). We assume it is given by a capital ratio as defined in section 2.1.1.

3.1 Conversion type

In the trigger event, CoCo's may be written down or partially or completely converted to equity. This is pre-specified in the contract. Let f be the conversion fraction, so that $f = 1$ corresponds to a complete conversion and $f = 0$ corresponds to no conversion (i.e, no change in liabilities). Let L be the book value of liabilities in the contract, and the conversion ratio R be the number of shares received for each bond. The conversion price is defined as:

$$C_p = \frac{fL}{R}$$

The conversion type can be specified by a given conversion ratio, or by a conversion price based on one of the following (De Spiegeleer and Schoutens 2011):

- **Price at trigger date**: The conversion price is the price of the share at the trigger event.

- **Price on issue**: The conversion price is the price of the share at the time of issue of the contract, or an average of a number of past prices at the time of issue.

- **Price with a floor**: The conversion price is the maximum of the price of the share at the trigger event and a certain pre-specified floor value.

3.2 Modelling CoCo's

With the previous specifications of the contract, CoCo's can be incorporated into the network model described in section 1 in a very similar way as bail-in, having the same network reconfiguration effects as depicted in figure 2.3. For simplicity, we consider the modelling of CoCo's independently of bail-in here, however they could easily be combined into one model. We assume that each bank has no more than one type of CoCo, which is consistent with past adoptions of CoCo's (De Spiegeleer and Schoutens 2011), although this assumption could be easily extended. We also consider only the case in which the conversion ratio is pre-specified in the contract, although this could be extended to a more general case in which the conversion ratio depends on the equity value at maturity.

It makes sense to place CoCo's into an own seniority class in the model. Let s_c be the seniority class of CoCo's, λ^C the vector of bank-specific CoCo conversion thresholds, R the vector of bank-specific conversion ratios, and f the vector of bank-specific conversion fractions. The conversion matrix C determines the amount of shares received by bank j if the CoCo issued by bank i is triggered in position $C_{i,j}$, and is defined by:

$$C_{i,j} = f_i * R_i * L_{i,j,s_c} \tag{3.1}$$

The calculation of the clearing payment matrix is similar to the bail-in case, described in section 2.2:

1. (*Initialization*) Start the iteration with $\bar{P}^1 = \bar{P}$, $\Theta^1 = \Theta$.
2. (*Begin Do-While*) Compute P^* according to the Elsinger algorithm using $\bar{P}^n$ and Θ^n.
3. Check if any CoCo trigger threshold has been breached.

   Λ^n is the diagonal matrix which indicates whether a bank has breached its CoCo threshold in the current iteration:

$$\Lambda^n = diag\left(\frac{e + (\Theta^n)' V^*(P^*) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s} - \sum_s \bar{P}^n_{\cdot,s}}{e + (\Theta^n)' V^*(P^*) + \sum_s \Pi'_{\cdot,\cdot,s} P^*_{\cdot,s}} < \lambda_C\right) \tag{3.2}$$

Γ^n keeps track of the banks who's CoCo's have triggered in some previous iteration:

$$\Gamma^n = diag\bigg(\max\left(\Lambda^1, \ldots, \Lambda^{n-1}\right)\bigg) \tag{3.3}$$

4. Update the liabilities vector and the holding matrix:

$$\bar{P}^{n+1}_{\cdot,s_c} = \Big(\mathbf{I} - \Lambda^n(\mathbf{I} - \Gamma^n)\Big)\bar{P}^n_{\cdot,s_c} + (1-f)\cdot\Lambda^n(\mathbf{I} - \Gamma^n)\bar{P}^n_{\cdot,s_c} \tag{3.4}$$

   The updated holding matrix is defined as follows. Let $\bar{c}_i = \sum_j C_{i,j}$. Then,

$$\Theta^{n+1} = \Big(\mathbf{I} - \Lambda^n(\mathbf{I} - \Gamma^n)\Big)\Theta^n + \Lambda^n(\mathbf{I} - \Gamma^n)\Big(diag\Big(1 - \bar{c}\Big) * \Theta^n + C'\Big) \tag{3.5}$$

5. (*Termination condition*) If $\Lambda^n(\mathbf{I} - \Gamma^n)\Big) = \mathbf{0}$, stop the iteration. Otherwise, continue to step 2.

Note that, by construction, each component of $\bar{P}_{\cdot,s_c}$ and each row of Θ are modified at most once during the iteration. Moreover, in each step of the iteration, one of these is modified, or else the iteration ends. Hence, the iteration is completed after at most n steps, where n is the number of banks in the network. Also by construction, when the iteration completes all CoCo's of banks that are under the CoCo threshold have triggered.

Combining the CoCo mechanism described above with the bail-in procedure described in section 2 would require checking whether CoCo's or bail-in are triggered first ($\lambda^B < \lambda^C$ or $\lambda^C < \lambda^B$) and then ensuring that in the first iteration where at least one of the two is breached, only the corresponding update operations will be effected. If $\lambda^C = \lambda^B$ then the update operations need to be performed in the same iteration, unless one assumes that e.g. the CoCo's will be converted first in this case as well.

Chapter 4

Ex-ante valuation

As pointed out by Barucca et al. 2016, the models of the previous sections correspond to a debt valuation at maturity (i.e, a calculation of payoffs). We now consider the problem of valuing debt before maturity in the previous framework, thus contributing to the research which began with Merton's structural model for pricing debt (Merton 1974), and was expanded by Suzuki 2002 and Fischer 2014, who adapted Merton's model to the interbank network framework.

Consider a time frame $\mathbb{T} = [t_0, T]$, where t_0 is the time at which the pricing is made, a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in\mathbb{T}}, \mathbb{P})$, and a n-dimensional Brownian motion W, such that the filtration $(\mathcal{F}_t)$ is generated by W. Assume that the external assets $e_t \in \mathbb{R}^{n\times\mathbb{T}}$ of each bank follow a diffusion process of the form:

$$\mathrm{d}e_t = \mu(t, e_t)\mathrm{d}t + \sigma(t, e_t)\mathrm{d}W_t \tag{4.1}$$

Where $\mu(t, e_t) \in \mathbb{R}^n$ is the vector of drifts, and $\sigma(t, e_t) \in \mathbb{R}^n$ is the vector of volatilities. For simplicity, we will restrict the model to the case where e_t is a multi-dimensional geometric Brownian motion, with $\mu(t, e_t) = \mu e_t$ and $\sigma(t, e_t) = \sigma e_t$ - however, more general models such as 4.1 could be considered without adding much more complexity. The external assets can be correlated, so assume W is a n-dimensional Brownian motion, with correlation matrix Σ. Under certain assumptions, this framework corresponds to a multidimensional Black-Scholes model. In particular, the following assumptions are made (see Merton 1974):

1. The external assets of each bank are assumed to be a trade-able asset in a market with no liquidity costs.

2. Similarly, liabilities $L_{i,j,s}$ and equity shares are assumed to be trade-able assets.

3. There is sufficient supply and demand in the market so that an investor can buy or sell all assets at any point.

4. It is possible to borrow and lend cash with the same rate of interest. The instantaneous rate of interest r is constant.

5. Short-sales are permitted for all assets.

6. All liabilities are to be paid at maturity, including interest rates (no coupon payments before maturity are considered).

4.1 Measurability of the payoff function

Assuming a fixed maturity T for all liabilities, consider the clearing payment vector function with bail-in as given in (2.8),

$$\Phi^{\mathrm{B}}(P, e_T)_{i,s} = \begin{cases} \bar{P}^B_{i,s}, & \text{if } (e_T)_i + \mathcal{A}^B_{i,s} \geq \bar{P}^B_{i,s} \\ ((e_T)_i + \mathcal{A}^B_{i,s})^+, & \text{otherwise} \end{cases}$$

(4.2)

Where the external assets are given by their value at maturity e_T. Define the payoff function as follows.

Definition 4.1.1 (Payoff function)**.** *The payoff function returns the clearing payment matrix for a given value of external assets,*

$$\Psi^B \colon \mathbb{R}^n \to \mathbb{R}^{n \times S} \tag{4.3}$$

$$e \mapsto \Psi^B(e). \tag{4.4}$$

Where $\Psi^B(e)$ *is the greatest fixed point of* $\Phi^B(\cdot, e)$*.*

Note that Ψ^B is bounded by $\bar{P}$, component-wise. Assuming that Ψ^B is measurable, the standard Black-Scholes theory can be applied to value the debt of each bank (see Lamberton and Lapeyre 2011 for a detailed introduction to stochastic calculus applied to finance). For example, the ex-ante valuation of the liability $L_{i,j,s}$ at time t_0 is given by:

$$V_{t_0} = e^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}_{t_0}[\Pi_{i,j,s} * \Psi^B(e_T)_{i,s}] \tag{4.5}$$

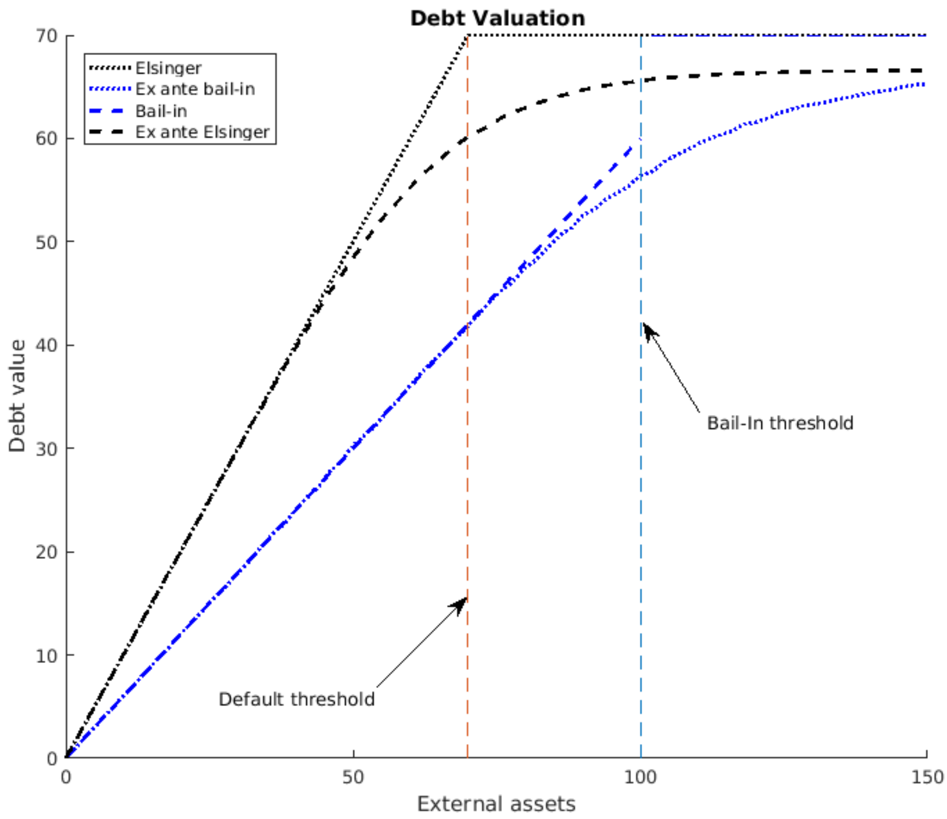
Where $\mathbb{Q}$ is the risk-neutral measure under which e follows the risk-neutral dynamics

$$\frac{\mathrm{d}e_t}{e_t} = r\mathrm{d}t + \sigma \mathrm{d}W^{\mathbb{Q}}_t \tag{4.6}$$

And $W^{\mathbb{Q}}$ is a Brownian motion with respect to $\mathbb{Q}$ and the natural filtration $(\mathcal{F}_t)$. The operator $\mathbb{E}_t[\cdot]$ indicates the conditional expectation with respect to $\mathcal{F}_t$. Note that in the valuation 4.5 we are assuming that the gained shares in case of bail-in don't contribute to the liabilities valuation. This assumption will be relaxed in the case of CoCo valuation in the next section.

Figure 4.1 shows the ex-ante valuation of the liabilities of a bank with only one seniority of liabilities $\bar{p} = 70$, recapitalization ratio $\lambda^R = 0.4$ and bail-in threshold $\lambda^B = 0.3$, similar to the case represented in figure 2.1. This example illustrates that debt payments at maturity are not concave with respect to the external assets at maturity, and that they are not a continuous function of the external assets at maturity.

Figure 4.1: Debt valuation of the example bank's liabilities. For the ex-ante valuation, the following model parameters were considered: $r = 0.05$, $\sigma = 0.3$, $T = 1$



As mentioned previously, a necessary condition for the value of debt (4.5) to be well defined is the measurability of Ψ^B. Fischer 2014 proves the measurability of the function with seniority structure - we extend this proof to the case of bail-in.

Theorem 4.1.1. *[Measurability of the payoff function] The function Ψ^B defined in (4.3) is measurable.*

Proof.

1. Firstly, consider the case of the clearing payment vector in a financial system with no seniority structure.

Theorem 1.2.1 states that the sequence $p^{i+1} = (W^*(p^i, e) + \bar{p})^+ \wedge \bar{p}$, with $p^0 = \bar{p}$, converges to the largest clearing payment vector. Lemma 1.2.1 states that the sequence $W^k(p, e)$ defined by $W^0(p, e) = e + \Pi' p - \bar{p}$ and $W^{k+1}(p, e) = W^0(p, e) + \Theta\Lambda^k W^{k+1}$, where $\Lambda^k = diag(W^k > \mathbf{0})$, converges to $W^*(p, e)$. Hence, defining $\Phi_W(W)$ to be a solution of the linear equation $\Phi_W(W) = W^0(p, e) + \Theta * diag(W > \mathbf{0}) * \Phi_W(W)$, $W^*(p, e)$ can be expressed as

$$W^*(p, e) = \lim_{k\to\infty} \Phi_W^k(W^0(p, e)) \tag{4.7}$$

The function Φ_W is a solution to a linear equation, hence it is measurable, and so is Φ_W^k by composing. Finally, $W^*(p, e)$ is measurable with respect to e almost surely, as it is a point-wise limit of measurable functions.

A similar argument can be made to show that the clearing payment vector $p^*(e)$ is a measurable function of e. Define $\Phi_p(p)$ by $\Phi_p(p) = (W^*(p, e) + \bar{p})^+ \wedge \bar{p}$. Because W^* is measurable with respect to e, so is Φ_p, and

$$p^*(e) = \lim_{k\to\infty} \Phi_p^k(\bar{p}) \tag{4.8}$$

is a measurable function of e.

2. Now, consider the case of seniority structure. The calculation of the clearing matrix with seniority structure was described through an iterative algorithm. Each step in the algorithm can be described as a function of the external assets e and the previous values of H. Moreover, in each iteration, all steps consist of transformations which are compositions of linear transformations, divisions and solutions to linear equations. All these are measurable. The algorithm has finite steps, so the resulting clearing matrix P^* is a measurable function of the input e.

3. The extension to bail-in is analogous. Each step of the algorithm for calculating the clearing matrix under bail-in consists of transformations which are compositions of linear transformations, divisions and solutions to linear equations, all of which are measurable with respect to the input e. A limiting argument analogous

to (4.8) proves the measurability of the resulting clearing payment matrix with respect to e.

□

4.2 Pricing CoCo's

The payoff of a CoCo is given by the total amount of liabilities if the trigger event has not occurred at maturity, and the value of the equity conversion if else. The payment function is given by:

$$\Phi^C(P, e_T)_{i,s} = \begin{cases} \bar{P}^C_{i,s}, & \text{if } (e_T)_i + \mathcal{A}^C_{i,s} \geq \bar{p}^C_{i,s} \\ ((e_T)_i + \mathcal{A}^C_{i,s})^+, & \text{otherwise} \end{cases}$$

(4.9)

Where $\bar{p}^C$ and $\mathcal{A}^C$ are as defined in section 3.2. The payoff function is defined as:

Definition 4.2.1 (Payoff function with CoCo's)**.** *The payoff function returns the clearing payment vector for a given value of external assets,*

$$\Psi^C : \mathbb{R}^n \to \mathbb{R}^{n \times S} \tag{4.10}$$

$$e \mapsto \Psi^C(e). \tag{4.11}$$

Where $\Psi^C(e)$ is the greatest fixed point of $\Phi^C(\cdot, e)$.

The proof of the measurability of Ψ^C is analogous to 4.1.1. Hence, the ex-ante valuation of the CoCo issued by bank i to bank j is:

$$V_{t_0} = e^{-r(T-t_0)}\mathbb{E}^{\mathbb{Q}}_{t_0}[\Pi_{i,j,s_c} * \Psi^C(e_T)_{i,s_c} + C_{i,j}V^*_i(e_T)1_{Conversion_i}] \tag{4.12}$$

Where $V^*(e)$ is the equity value of each bank after contagion,

$$V^*(e) = \Big(e + \sum_{s=1}^{S} \Pi'_{\cdot,\cdot,s}\, \Psi^C(e)_{\cdot s} - \sum_{s=1}^{S} \bar{P}^C_{\cdot,s} + (\Theta^C)'V^*(e)\Big)^+$$

And $1_{Conversion_i}$ indicates whether the CoCo of bank i has triggered. The value $C_{i,j}V_i^*(e_T)1_{Conversion_i}$ in the expectation corresponds to the value of the shares gained by the buyer of the CoCo if it is converted.

Conclusions

In this paper we have presented a model of network valuation for different financial contracts, namely equity participations, debt liabilities of different seniority levels and contingent convertible debt instruments. The starting point of our model is a combination of the work of Elsinger 2009, who model the valuation of equity and debt of different seniority levels, with the work of Barucca et al. 2016 and Fischer 2014, who discuss the network valuation of debt at time points before maturity under stochastic prices for external assets. The combined model can be seen as an extension of the model of Merton 1974 to the multidimensional case with cross-holdings of different instruments. We extend the combined model to include contingent convertible debt instruments as well as the bail-in of entire seniority classes of financial instruments to reach capitalization targets. We provide a Matlab implementation of the model, in the hope of providing useful resources for policy makers.

The model presented herein provides a framework for assessing the risk of contagion in financial systems, and the potential impact of policy measures, in particular with regards to bail-in decisions. While the model is of theoretical concern, we demonstrate its applicability by laying out an example implementation adapted to current capital definitions and bail-in regulations in the EU. Adaptions to other regulatory environments consist largely in an appropriate mapping of available data to the variables and parameters of the model. Potential outputs of the model in the context of systemic stress tests include: (i) loss 'add-ons' to microprudential stress tests to capture the effects of contagion. (ii) Rankings of financial institutions by systemic contagiousness and vulnerability by considering idiosyncratic defaults of individual institutions or subgroups. (iii) Merton 1974-type default probabilities and loss rates for banks. Furthermore, (iv) the model allows assessing the sensitivity of all of the aforementioned outputs to bail-in decisions and CoCo parameters.

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Appendices

Appendix A

Conversion matrix properties

Lemma A.0.1.

(i) When Equity$_j^n$ + Bail-In$_j^n \leq 0$, any bail-in is non-positive.

(ii) When Equity$_j^n \leq 0$, Equity$_j^n$ + Bail-In$_j^n \geq 0$ and $f_i^p = \sum_{j,s}\left(C_{i,j,s}^{\text{fair, } y=1} - C_{i,j,s}\right)$, $\forall i$, a fair conversion matrix implies a non-positive bail-in.

(iii) When Equity$_j^n \geq 0$ and $f_i^p = \sum_{j,s}\left(C_{i,j,s}^{\text{fair, } y=1} - C_{i,j,s}\right)$, $\forall i$, a fair conversion matrix implies a neutral bail-in.

Proof. (i) follows trivially from definition 2.2.2 by noting that Bail-In$_j^n \geq 0$ for all j, n.

We then note that when $f_j^p = \sum_{i,s}\left(C_{j,i,s}^{\text{fair, } y=1} - C_{j,i,s}\right)$ we obtain:

$$\Theta_{j,i}^{n+1} = \left(1 - \sum_i \sum_s C_{j,i,s}^{\text{fair, } y=1}\right)\Theta_{j,i}^n + \sum_s C_{j,i,s} \tag{A.1}$$

We now consider the case Equity$_j^n \leq 0$, Equity$_j^n$ + Bail-In$_j^n \geq 0$ (ii). We insert Eq. (A.1) into the definition of a non-positive bail-in and obtain:

$$\begin{aligned}
\forall\, i, j :\ & \left(\left(1 - \sum_i \sum_s C_{j,i,s}^{\text{fair, } y=1}\right)\Theta_{j,i}^n + \sum_s C_{j,i,s}^{\text{non-positive}}\right)\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right) && \text{(A.2)}\\
& \leq \sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n \\
\iff \forall\, i, j :\ & \sum_s C_{j,i,s}^{\text{non-positive}}\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right) && \text{(A.3)}\\
& + \Theta_{j,i}^n\left(1 - \sum_i \sum_s C_{j,i,s}^{\text{fair, } y=1}\right)\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right) \leq \sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n && \text{(A.4)}
\end{aligned}$$

If $\text{Bail-In}_j^n = 0$, the result clearly holds. Assume $\text{Bail-In}_j^n > 0$, and note that

$$\forall\ j:\ \left(1 - \sum_i \sum_s C_{j,i,s}^{\text{fair},\, y=1}\right)\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right) = \tag{A.5}$$

$$\left(1 - \frac{\sum_i \sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n}{\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right)}\right)\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right) = \tag{A.6}$$

$$\text{Equity}_j^n + \text{Bail-In}_j^n - \sum_s \sum_i \Pi_{j,i,s}\text{Bail-In}_{j,s}^n = \text{Equity}_j^n \leq 0 \tag{A.7}$$

and obtain by inserting (A.5) into (A.2):

$$\forall\ i,j:\ \sum_s C_{j,i,s}^{\text{non-positive}}\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right) + \Theta_{j,i}^n\left(\text{Equity}_j^n\right) \leq \sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n \tag{A.8}$$

Setting $C^{\text{non-positive}} = C^{\text{fair}}$, which in this case is given by y^{fair} (Eq. 2.5), yields:

$$\forall\ i,j:\ \gamma\frac{\sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n}{\text{Bail-In}_j^n}\left(\text{Equity}_j^n + \text{Bail-In}_j^n\right) + \Theta_{j,i}^n\left(\text{Equity}_j^n\right) \leq \sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n \tag{A.9}$$

$$\Longleftrightarrow \forall\ i,j:\ \gamma\frac{\sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n}{\text{Bail-In}_j^n}\left(\text{Equity}_j^n\right) + \gamma\sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n + \Theta_{j,i}^n\left(\text{Equity}_j^n\right) \tag{A.10}$$
$$\leq \sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n$$

$$\Longleftrightarrow \forall\ i,j:\ \gamma\frac{\sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n}{\text{Bail-In}_j^n}\left(\text{Equity}_j^n\right) + (\gamma - 1)\sum_s \Pi_{j,i,s}\text{Bail-In}_{j,s}^n + \Theta_{j,i}^n\left(\text{Equity}_j^n\right) \leq 0 \tag{A.11}$$

which is strictly less than zero in the case that $\text{Bail-In}_j^n > 0$ and $\text{Equity}_j^n \leq 0$.

We now consider the case $\text{Equity}_j^n, \text{Bail-In}_j^n \geq 0$ (iii). We note that in this case $C^{\text{fair},\, y=1} = C^{\text{fair}}$ and insert Eq. (A.1) into the definition of a neutral bail-in to obtain:

$$\forall\ i,j:\ \left(\left(1-\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\Theta^n_{j,i}+\sum_s C^{\text{neutral}}_{j,i,s}\right)\left(\text{Equity}^n_j+\text{Bail-In}^n_j\right)-\Theta^n_{j,i}\left(\text{Equity}^n_j\right) \tag{A.12}$$

$$=\sum_s \Pi_{j,i,s}\text{Bail-In}^n_{j,s} \tag{A.13}$$

$$\Longleftrightarrow \sum_s C^{\text{neutral}}_{j,i,s}\left(\text{Equity}^n_j+\text{Bail-In}^n_j\right)+\left(1-\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\Theta^n_{j,i}\left(\text{Bail-In}^n_j\right)- \tag{A.14}$$

$$\left(\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\Theta^n_{j,i}\left(\text{Equity}^n_j\right)=\sum_s \Pi_{j,i,s}\text{Bail-In}^n_{j,s} \tag{A.15}$$

$$\Longleftrightarrow \Theta^n_{j,i}\left(\left(1-\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\left(\text{Bail-In}^n_j\right)-\left(\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\left(\text{Equity}^n_j\right)\right)+ \tag{A.16}$$

$$\sum_s C^{\text{neutral}}_{j,i,s}\left(\text{Equity}^n_j+\text{Bail-In}^n_j\right)=\sum_s \Pi_{j,i,s}\text{Bail-In}^n_{j,s} \tag{A.17}$$

$$\Longleftrightarrow \sum_s C^{\text{neutral}}_{j,i,s}=\frac{\sum_s \Pi_{j,i,s}\text{Bail-In}^n_{j,s}}{\text{Equity}^n_j+\text{Bail-In}^n_j}=\sum_s C^{\text{fair}}_{j,i,s} \tag{A.18}$$

$$\Longleftarrow C^{\text{neutral}}_{j,i,s}=C^{\text{fair}}_{j,i,s} \tag{A.19}$$

where the last step follows from:

$$\forall\ j:\ \left(1-\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\left(\text{Bail-In}^n_j\right)-\left(\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\left(\text{Equity}^n_j\right)= \tag{A.20}$$

$$\text{Bail-In}^n_j-\left(\sum_i\sum_s C^{\text{fair}}_{j,i,s}\right)\left(\text{Equity}^n_j+\text{Bail-In}^n_j\right)= \tag{A.21}$$

$$\text{Bail-In}^n_j-\left(\sum_s\sum_i \frac{\Pi_{j,i,s}\text{Bail-In}^n_{j,s}}{\text{Equity}^n_j+\text{Bail-In}^n_j}\right)\left(\text{Equity}^n_j+\text{Bail-In}^n_j\right)= \tag{A.22}$$

$$\text{Bail-In}^n_j-\sum_s\sum_i \Pi_{j,i,s}\text{Bail-In}^n_{j,s}=0 \tag{A.23}$$

since $\sum_i\sum_s \Pi_{j,i,s}=0 \Leftrightarrow \sum_s(\bar{P}^n_{\cdot,s})_j=0 \Leftrightarrow \text{Bail-In}^n_j=0$ for all j,n.

□

Appendix B

Matlab Code

B.1 Elsinger algorithm

```
%% calcElsinger
% Computes a clearing payment vector for a given financial system,
    using
% the Elsinger (2009) methodology.
%
% *Inputs*
%
% * vecE: vector (banks x 1) of other assets
% * matL: matrix (banks x banks) of interbank claims
% * matTheta: matrix (banks x banks) of relative interbank holdings
%
% *Outputs*
%
% * vecP: clearing payment vector
% * vecEquity: equity values after contagion
%
% Author: Supervisor of dissertation candidate
% Last modified: 18.06.2018
%


function [vecP, vecEquity] = calcElsinger(vecE, matL, matTheta)

%%
% Variable initialisations
vecPbar = sum(matL,2);
matPi = matL ./ repmat(vecPbar,1,length(matL(:,1)));
matPi(isnan(matPi)) = 0;

% Convergence parameters
```

```
dblPrecision = max(vecPbar)/100000;
numMaxIterations=100;

%% Loop to find fixed point
% Initialise loop variables
vecP = vecPbar;
blnLoop = true;
numIterations=0;

while blnLoop
    vecP_old = vecP;

    %%%
    % Compute new equity value
    vecEquity = calcEquityValue(vecE,matPi,vecP,vecPbar,matTheta);
    posDefaulted = vecEquity < 0;

    %%%
    % Compute new p vector
    vecEquity = max(vecEquity,0);
    vecAux = matPi'*vecP+vecE+matTheta'*vecEquity;
    vecP(posDefaulted) = max(0,vecAux(posDefaulted));

    %%%
    % Check for convergence
    blnLoop = norm(abs(vecP-vecP_old)) > dblPrecision;
    if numIterations>numMaxIterations
        blnLoop=false;
        disp('No convergence in calcElsinger');
    end
    numIterations=numIterations+1;
end

end

%% Sub-function calcEquityValue
function vecEquity = calcEquityValue(vecE,matPi,vecP,vecPbar,matTheta
    )

%% Declarations
% Convergence parameters
dblPrecision = max(vecPbar)/100000;
numMaxIterations=100;
%%%
% Variable initialisations
vecEquity=matPi'*vecP+vecE-vecPbar;

%% Loop to find fixed point
% Initialise loop variables
```

```
blnLoop=true;
numIterations=0;
if sum(sum(matTheta))~=0
    while blnLoop
        vecEquity_old=vecEquity;
        %%%
        % Compute new equity vector
        vecEquity = max(vecEquity,0);
        vecEquity = vecE + matPi'*vecP - vecPbar + matTheta'*
    vecEquity;

        %%%
        % Check for convergence
        blnLoop = norm(abs(vecEquity-vecEquity_old)) > dblPrecision;
        if numIterations>numMaxIterations
            blnLoop=false;
            disp('No convergence in calcEquityValue');
        end
        numIterations=numIterations+1;
    end
end

end
```

B.2 Elsinger algorithm with seniority structure

```
%% calcElsingerSeniority
% Computes a clearing payment matrix for a given financial system,
    using
% the Elsinger 2009 methodology with seniority structure.
%
% *Inputs*
%
% * vecE: vector (banks x 1) of other assets
% * matL: matrix (banks x banks x seniorities) of interbank claims
% * matTheta: matrix (banks x banks) of interbank holdings

%
% *Outputs*
%
% * matP: clearing payment matrix (banks * seniorities)
% * vecEquityAfterContagion: equity values after contagion
% * matTheta: matrix (banks x banks x seniorities) of interbank
    holdings
% * vecDefaultedBanks: boolean vector (banks x 1), 1 if bank has
    defaulted

% Authors: Dissertation candidate and supervisor
```

```
% Last modified: 18.06.2018
%

function [matP, vecEquity, matTheta, vecDefaultedBanks] =
    calcElsingerSeniority(vecE,matL,matTheta)

% Define Elsinger 2009 (seniority) variables

numSeniority = size(matL);
numSeniority = numSeniority(3);
numBanks = length(vecE);
vecDefaultedBanks = false(numBanks,1);
vecH = ones(numBanks,1) * numSeniority;
blnloop = true;
matLH = zeros(numBanks);
matPi = matL;
matPbar = zeros(numBanks,numSeniority);
for s=1:numSeniority
    matPbar(:,s) = matPbar(:,s) + sum(matL(:,:,s),2);
end
for s=1:numSeniority
    matPi(:,:,s) = matL(:,:,s) ./ repmat(matPbar(:,s),1,numBanks);
end
matPi(isnan(matPi)) = 0;

%% Compute clearing payment vector
% Use Elsinger 2009 algorithm with seniority structure

while blnloop
    % define vector vecEH
    vecEH = vecE;
    for i=1:numBanks
        for j=1:numBanks
            for s = 1:(vecH(j)-1)
                vecEH(i) = vecEH(i) + matPi(j,i,s)*matPbar(j,s);
            end
        end
    end

    for i=1:numBanks
        for s = 1:(vecH(i)-1)
            vecEH(i) = vecEH(i) - matPbar(i,s);
        end
    end

    % define matrix matLH
    for i=1:numBanks
        for j = 1:numBanks
            matLH(i,j) = matL(i,j,vecH(i));
```

```
        end
    end


    [vecP, vecEquity] = calcElsinger(vecEH, matLH, matTheta);
    blnDefault = vecEquity < 0 & vecH>1;
    vecDefaultedBanks = vecDefaultedBanks | blnDefault;

    if (blnDefault == zeros(numBanks,1))
        blnloop = false;
    end


    % update variables
    vecH = vecH - blnDefault;

matP = zeros(numBanks, numSeniority);

for i=1:numBanks
    for s = 1:numSeniority
        if(vecH(i) > s)
            matP(i,s) = matPbar(i,s);
        end
        if(vecH(i) == s)
            matP(i,s) = vecP(i);
        end
        if(vecH(i) < s)
            matP(i,s) = 0;
        end
    end
end

% compute equity vector

vecEquity = vecE;
for s=1:numSeniority
    vecEquity = vecEquity + matPi(:,:,s)'*matP(:,s) - matPbar(:,s);
end

vecEquity = vecEquity + max(0,matTheta' * vecEquity);




end
```

B.3 Clearing payment matrix with bail-in

```
%% calcElsingerBailIn
% Computes a clearing payment vector for a given financial system, using
% the Elsinger 2009 methodology with seniority structure and bail-in.
%
% *Inputs*
%
% * vecE: vector (banks x 1) of other assets
% * vecLambdaB: vector (banks x 1) of bail-in thresholds
% * vecLambdaR: vector (banks x 1) of recapitallization thresholds
% * matL: matrix (banks x banks x seniorities) of interbank claims
% * matTheta: matrix (banks x banks) of interbank holdings
% * numK: integer, determines the number of seniority classes corresponding to bail-in
% * funConversion: function that returns the desired conversion factor.
%                  Inputs: MatBailIn - matrix (banks x numK) of bail-in amount per bank and seniority class
%                          vecEquity: equity vector (banks x 1)
%                          matPi: matrix (banks x banks x seniorities) of
%                          relative liabilities
%                  Outputs: conversion matrix (banks x banks x numK)
%
% *Outputs*
%
% * matP: clearing payment matrix (banks * seniorities)
% * vecEquityAfterContagion: equity values after contagion
% * matTheta: matrix (banks x banks x seniorities) of interbank holdings
% * vecDefaultedBanks: boolean vector (banks x 1), 1 if bank has defaulted
% * vecBailedInBanks: boolean vector (banks x 1), 1 if bank has been
%                     bailed-in
% Authors: Dissertation candidate and supervisor
% Last modified: 18.06.2018
%

function [matP, vecEquity, matTheta, vecDefaultedBanks, vecBailedInBanks] = calcElsingerBailIn(vecE,matL, matTheta, numK, funConversion, vecLambdaB, vecLambdaR)


% Define Elsinger 2009 (seniority) variables

numSeniority = size(matL);
numSeniority = numSeniority(3);
numBanks = length(vecE);
vecDefaultedBanks = false(numBanks,1);
```

```
vecBailedInBanks = false(numBanks,1);
blnLoop = true;
numIterations = 0;
matF = zeros(numBanks, numBanks, numSeniority); % gained shares
vecBailInAble = zeros(numBanks,1);
matPi = matL;
matPbar = zeros(numBanks,numSeniority);
matBailIn = zeros(numBanks, numK);


for s=1:numSeniority
    matPbar(:,s) = matPbar(:,s) + sum(matL(:,:,s),2);
end
for s=1:numSeniority
    matPi(:,:,s) = matL(:,:,s) ./ repmat(matPbar(:,s),1,numBanks);
end
matPi(isnan(matPi)) = 0;

% Convergence parameters
dblPrecision = max(matPbar)/100000;
numMaxIterations=100;

%% Compute clearing payment vector
% Use Elsinger 2009 algorithm with seniority structure

while blnLoop
    [matP, vecEquity, matTheta, vecDefaultedBanks] = calcElsingerSeniority(vecE,matL,matTheta);

    matPbarOld = matPbar;
    % bail-in-able liabilities total

    for s=(numSeniority -numK+1):numSeniority
        vecBailInAble = vecBailInAble + matPbar(:,s);
    end
    vecLambda = vecEquity ./ (vecEquity + sum(matPbar,2)); % vector of capital ratios
    vecBailIn = max(0,sum(matPbar,2) - (1 - vecLambdaR).*(vecEquity + sum(matPbar,2)));
    vecBailIn(vecLambda > vecLambdaB) = 0; % bail in only if lambda < lambdaB
    vecDefaultedBanks = vecDefaultedBanks | (vecBailIn>0 & vecBailInAble==0);
    vecBailIn = min(vecBailInAble, vecBailIn);
    vecBailedInBanks = vecBailedInBanks | (vecBailIn>0);

    % bail-in process

    for s=(numSeniority-numK+1):numSeniority
```

```
        vecJunior = zeros(numBanks,1);
        for s2=(s+1):numSeniority
            vecJunior = vecJunior + matPbar(:,s2);
        end
        for i=1:numBanks
            if vecBailIn(i) >= vecJunior(i)
                matPbar(i,s) = max(0,(matPbar(i,s)- vecBailIn(i) + vecJunior(i)));
                matBailIn(i,s + numK - numSeniority) = matPbarOld(i,s) - matPbar(i,s);
            end

        end
    end

    matConversion = funConversion(matBailIn, vecEquity, matPi);
    for i=1:numBanks
            for j=1:numBanks
                for s=(numSeniority -numK+1):numSeniority
                  matF(i,j,s) =  matConversion(i,j,s + numK - numSeniority);
                end
            end
    end

    vecFbar = sum(sum(matF, 3),3);



    % update liabilities array

    for s=1:numSeniority
        matL(:,:,s) = matPi(:,:,s).*matPbar(:,s);
    end

    % update holding matrix

    for i=1:numBanks
        for j=1:numBanks
            matTheta(j,i) = (1 - vecFbar(i))*matTheta(j,i) + sum(matF(j,i,:));
        end
    end

    blnLoop = norm(sum(abs(matPbarOld-matPbar),2)) > dblPrecision;
   if numIterations>numMaxIterations
       blnLoop=false;
       disp('No convergence in calcElsingerBailIn');
   end
```

```
        numIterations=numIterations+1;
end


end
```